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Squeezed States for General Multiphoton Systems: Towards the Displacement-operator Formalism

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Abstract

We propose a displacement-operator approach to some aspects of squeezed states for general multiphoton systems. The explicit displacement-operators of the squeezed vacuum and the coherent states are achieved and expresses as the ordinary exponential form. As a byproduct the coherent states of the q-oscillator are obtained by the *usual exponential* displacement-operator.

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1 Introduction

It is well known that there are three equivalent definitions of the coherent states of the harmonic oscillators, that is, (1) the displacement-operator acting on the vacuum states, (2) the eigenstates of the annihilation operator and (3) the minimum uncertainty states. Generalizations to the coherent states of arbitrary Lie groups were extensively studied in the literature [1, 2]. However, their extension to the squeezed states [3], which became more and more interesting in quantum optics [4] and gravitational wave detection[5], gave the equivalent results only for the harmonic oscillator. The minimum uncertainty method works well for both the coherent and squeezed states for any symmetry systems [6, 7] and the ladder-operator squeezed states for general systems are described in [8]. Both methods are closely connected as discussed in [8]. Now the exception is that there is no general approach to the displacement-operator squeezed states for the general systems although there are some works toward this goal [9].

In this letter we shall investigate squeezed states for the general multiphoton multimode systems (see Eqs.(2), (7) and (47)). We first formulate the algebras of multiphoton creation and annihilation operators, and then discuss the squeezed states from the ladder-operator definition. For the zero squeezing case, we derive the explicit coherent state, which is written as an ordinary exponential displacement-operator acting on the vacuum state. The squeezed vacuum of this system is also explicitly obtained as an exponential displacement-operator, which is the *squeeze operator*, acting on the vacuum state. However, the product of coherent and the squeeze operators does not bring us the squeezed states equivalent to those obtained by the ladder-operator method. As a byproduct, the coherent states of the q-deformed oscillator [11] are obtained and expressed as the usual exponential displacement-operator acting on the vacuum states.

2 Multiphoton algebra

We first recall the ladder-operator approach of Nieto and Truax. They describe that *the general ladder-operator squeezed states are the eigenstates of a linear combination of the lowering and raising operators* [8]

$$\left(\mu A + \nu A^{\dagger}\right)|\beta\rangle = \beta|\beta\rangle, \tag{1}$$

where μ and ν are complex constants, A and its hermitian conjugate $A^{\dagger}$ are the lowering (annihilation) and raising (creation) operators, respectively. They satisfy the commutation relation

$$\left[A, A^{\dagger}\right] = C, \tag{2}$$

in which C is an operator to be specified later. We consider the eigenvalue equation (1) in the parameter region $|z| = |-\nu/\mu| < 1$. This means that the operator $\mu A + \nu A^{\dagger}$ has more annihilation than creation operators (see (24)). Here we consider only the single mode case and generalization to multimode case will be discussed in section 4. In quantum optics A is usually the multiphoton lowering operator in the form

$$A = f(N)a^{m}, \tag{3}$$

where a and $a^{\dagger}$ are the annihilation and creation operators of the photon satisfying $\left[a, a^{\dagger}\right] = 1$, $N = a^{\dagger}a$, and m is a positive integer. As usual the Fock states of the oscillator a and

$a^\dagger$ are denoted by $|n\rangle$, $n = 0, 1, \ldots$, $a|0\rangle = 0$, $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. The function $f(N)$ specifies the intensity dependent coupling, which is in general complex and we assume that $f(x)$ does not have zeros at non-negative integer values of x. By using $aa^\dagger = N+1$, $a^2(a^\dagger)^2 = (N+1)(N+2)$, etc, we obtain

$$\begin{aligned} AA^\dagger &= (N+1)(N+2)\cdots(N+m)f(N)f^*(N), & (4)\\ A^\dagger A &= (N-m+1)(N-m+2)\cdots Nf(N-m)f^*(N-m). & (5) \end{aligned}$$

It is obvious that we only need to restrict our discussions to the sector S_i $(i = 0, 1, \cdots, m-1)$ spanned by the Fock states $|nm+i\rangle$ (n non-negative integers). Introducing the *multiphoton number operator* $\mathcal{N}_i \equiv \mathcal{N}$ $(i = 0, 1, \cdots, m-1)$ in the sector S_i

$$\mathcal{N} = \frac{1}{m}(N-i), \quad i = 0, 1, \cdots, m-1, \tag{6}$$

and $F(\mathcal{N}+1) \equiv (m\mathcal{N}+1+i)\cdots(m\mathcal{N}+m+i)f(m\mathcal{N}+i)f^*(m\mathcal{N}+i)$, we can recast the system (2), (4), (5) in the following form

$$\begin{aligned} &AA^\dagger = F(\mathcal{N}+1), \quad A^\dagger A = F(\mathcal{N}),\\ &\left[\mathcal{N}, A^\dagger\right] = A^\dagger, \quad [\mathcal{N}, A] = -A. \end{aligned} \tag{7}$$

These are the starting relations of this letter, which we call *general intensity dependent m-photon algebra.* [3] With these definitions, the operator C in (2) is $C = F(\mathcal{N}+1) - F(\mathcal{N})$. Note that the r.h.s. of (5) vanishes on the Fock states $|n\rangle$ for $0 \le n \le m-1$, which implies $F(0) = 0$ in each sector.

The system (7) is general enough to cover many interesting examples: The case $m = 1$ and $f(N) = 1$ is the harmonic oscillator; for $m = 1$ and $f(N) = \sqrt{2k+N}$ $(k \ge 0)$ we recover the Holstein-Primakoff realization of the $su(1,1)$ algebra [12] by identifying

$$K^+ \equiv A^\dagger, \qquad K^- \equiv A, \qquad K^0 \equiv k + N. \tag{8}$$

The square realization of $su(1,1)$ corresponds to: $m = 2$ and $f(N) = 1/2$

$$\begin{aligned} K^+ &\equiv \frac{1}{2}(a^\dagger)^2, \quad K^- \equiv \frac{1}{2}a^2,\\ K^0 &\equiv \frac{1}{2}\left(N + \frac{1}{2}\right) = \begin{cases} \mathcal{N}_0 + \frac{1}{4} & \text{in } S_0,\\ \mathcal{N}_1 + \frac{3}{4} & \text{in } S_1. \end{cases} \end{aligned} \tag{9}$$

Both realizations (8) and (9) satisfy the $su(1,1)$ defining relations

$$\begin{aligned} \left[K^+, K^-\right] &= -2K^0, \quad \left[K^0, K^+\right] = K^+,\\ \left[K^0, K^-\right] &= -K^-. \end{aligned} \tag{10}$$

It is convenient to introduce an orthonormal basis for S_i

$$\|n\rangle = \frac{1}{\sqrt{F(n)!}}\left(A^\dagger\right)^n \|0\rangle, \tag{11}$$

[3] A similar algebra appeared in Heisenberg's theory of non-linear spinor dynamics [14], Eq.(9.6). We thank D. Fairlie for calling our attention to this paper.

where $\|0\rangle = |i\rangle$ is the vacuum state of sector S_i satisfying $A\|0\rangle = \mathcal{N}\|0\rangle = 0$ and $F(n)! \equiv F(n)F(n-1)\cdots F(1), \quad F(0)! \equiv 1$. On this basis we have

$$\begin{aligned} A^\dagger\|n\rangle &= \sqrt{F(n+1)}\|n+1\rangle, \\ A\|n\rangle &= \sqrt{F(n)}\|n-1\rangle, \\ \mathcal{N}\|n\rangle &= n\|n\rangle. \end{aligned} \tag{12}$$

It is very tempting to apply the idea of the system (7) to the multiphoton (m-photon) coherent states of the q-deformed oscillator. Let us choose (q is a *real* deformation parameter)

$$f(N) \equiv \left\{ \frac{1}{(N+1)\cdots(N+m)} \left[\frac{N}{m}+1\right] \right\}^{\frac{1}{2}}, \tag{13}$$

where $[x] \equiv (q^x - q^{-x})/(q-q^{-1})$, and define

$$\begin{aligned} &b_q \equiv A = f(N)a^m, \quad b_q^\dagger \equiv A^\dagger = (a^\dagger)^m f(N), \\ &N_q \equiv \mathcal{N} + \frac{i}{m}. \end{aligned} \tag{14}$$

Then by using (4) and (5) we would obtain *formally* the following relations

$$\begin{aligned} b_q b_q^\dagger &= \frac{(N+1)(N+2)\cdots(N+m)}{(N+1)(N+2)\cdots(N+m)}[N_q+1] \\ &= [N_q+1], \end{aligned} \tag{15}$$

$$\begin{aligned} b_q^\dagger b_q &= \frac{(N-m+1)(N-m+2)\cdots N}{(N-m+1)(N-m+2)\cdots N}[N_q] \\ &= [N_q], \end{aligned} \tag{16}$$

$$[N_q, \quad b_q^\dagger \quad] = b_q^\dagger, \qquad [N_q, b_q] = -b_q. \tag{17}$$

Eqs.(15),(16),(17) are in fact a multiphoton realization of the q-oscillator,

$$b_q b_q^\dagger - q b_q^\dagger b_q = q^{-N_q}. \tag{18}$$

(The case $m=1$ was found in [15]. See also [16], [17] and [18].) It should be remarked that the eigenvalues of N_q are not integers except for the $i=0$ sector.

By close inspection, however, one finds that the relation (16)

$$b_q^\dagger b_q = [N_q]$$

is not true on the vacuum in each sector S_i ($i \geq 1$ and $m > 1$). Obviously the vacuum of the i-th sector $\|0\rangle = |i\rangle$ vanishes when applied by b_q,

$$b_q\|0\rangle = f(N)a^m|i\rangle = 0, \quad i = 0, 1, \ldots, m-1. \tag{19}$$

On the other hand, as remarked above, $[N_q]\|0\rangle = [\frac{i}{m}]\|0\rangle$ is *non-vanishing* for $i \geq 1$. This apparent inconsistency is caused by $0/0 = 1$ in (16), since $N-i$ in the numerator and denominator vanish on $\|0\rangle = |i\rangle$. To sum up, the relations (18) and (16) are broken only by the vacuum expectation value and all the other relations are correct. It would be very interesting if one could find physical applications of the "spontaneously broken" *q-deformed multi-photon coherent states.* [4]

[4]If we introduce the intensity and sector-dependent multiphoton coupling then we can obtain the q-deformed oscillator in each sector. Namely, if we define $a_q = \sqrt{\frac{[\mathcal{N}+1]}{(N+1)\cdots(N+m)}}\, a^m$ in each sector, then it is easy to see that $a_q a_q^\dagger = [\mathcal{N}+1]$ and $a_q^\dagger a_q = [\mathcal{N}]$ are satisfied as operator equations.

3 Squeezed states

Now we shall solve the eigenvalue equation (1). We expand the state $|\beta\rangle$ as

$$|\beta\rangle = \sum_{n=0}^{\infty} C_n \| n\rangle, \tag{20}$$

Then equation (1) leads to the following recursion relations

$$\begin{aligned} C_{n+1} &= z\sqrt{\frac{F(n)}{F(n+1)}}C_{n-1} + \frac{\beta}{\mu\sqrt{F(n+1)}}C_n, \\ C_1 &= \frac{\beta}{\mu\sqrt{F(1)}}C_0, \end{aligned} \tag{21}$$

where $z = -\nu/\mu$. We have not been able to obtain a closed expression of C_n for the general case. We now consider some special cases.

3.1 Squeezed vacuum and squeeze operator

Let us first consider the squeezed vacuum $|v\rangle$ annihilated by $\mu A + \nu A^\dagger$

$$\left(\mu A + \nu A^\dagger\right)|v\rangle = 0, \tag{22}$$

and express it in terms of an exponential displacement-operator (squeeze operator) acting on the vacuum state. In this case with $\beta = 0$, the C_n's are easily obtained as

$$C_{2k+1} = 0, \quad C_{2k} = C_0 z^k \sqrt{\frac{F(2k-1)!!}{F(2k)!!}}, \tag{23}$$

where $F(2k)!! = F(2k)F(2k-2)\cdots F(2)$, $F(2k-1)!! = F(2k-1)F(2k-3)\cdots F(1)$ and $F(0)!! = F(-1)!! \equiv 1$. Then we have

$$\begin{aligned} |v\rangle &= C_0 \sum_{k=0}^{\infty} z^k \sqrt{\frac{F(2k-1)!!}{F(2k)!!}} \| 2k\rangle \\ &= C_0 \sum_{k=0}^{\infty} z^k \frac{\left(A^{\dagger 2}\right)^k}{F(2k)!!} \| 0\rangle. \end{aligned} \tag{24}$$

It is easy to check that the above infinite series converges if $|z| < 1$ under mild assumptions on the asymptotic behavior of $f(x)$, e.g., $f(x) \simeq x^\alpha$ for $x \to \infty$. By making use of the following identity

$$\left(\frac{\mathcal{N}}{F(\mathcal{N})}A^{\dagger 2}\right)^k = \left(A^{\dagger 2}\right)^k \frac{\mathcal{N}+2}{F(\mathcal{N}+2)} \cdots \frac{\mathcal{N}+2k}{F(\mathcal{N}+2k)}, \tag{25}$$

we can rewrite (24) as

$$\begin{aligned} |v\rangle &= C_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{z}{2}A^{\dagger 2}\right)^k \frac{(\mathcal{N}+2)\cdots(\mathcal{N}+2k)}{F(\mathcal{N}+2)\cdots F(\mathcal{N}+2k)} \| 0\rangle \\ &= C_0 \exp\left(\frac{z\mathcal{N}}{2F(\mathcal{N})}A^{\dagger 2}\right) \| 0\rangle. \end{aligned} \tag{26}$$

Following the terminology of the oscillator, the operator

$$S(z) = C_0 \exp\left(\frac{z\mathcal{N}}{2F(\mathcal{N})} A^{\dagger 2}\right) \tag{27}$$

is referred to as the *generalized* squeeze operator.

3.2 Multiphoton coherent states

Next let us consider the case $\nu = z = 0$. In this case the equation (1) reduces to the eigenvalue equation of the annihilation operator A and the resulting states are the coherent states. From Eq.(21) with $z = 0$ we can easily obtain the C_n's as

$$C_n = \frac{C_0 \alpha^n}{\sqrt{F(n)!}}, \tag{28}$$

where $\alpha = \beta/\mu$. Then the eigenstate $|\beta/\mu\rangle \equiv |\alpha\rangle$ is obtained as

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{F(n)!} (A^\dagger)^n \|0\rangle. \tag{29}$$

In terms of the so-called deformed exponential function $\exp_F(x) = \sum_{n=0}^{\infty} \frac{x^n}{F(n)!}$ one can express the coherent state as $|\alpha\rangle = C_0 \exp_F(\alpha A^\dagger)\|0\rangle$. Here we would rather like to use the usual exponential displacement-operator. To do this, we use the following relation

$$\left(\frac{\mathcal{N}}{F(\mathcal{N})} A^\dagger\right)^n = (A^\dagger)^n \frac{\mathcal{N}+1}{F(\mathcal{N}+1)} \cdots \frac{\mathcal{N}+n}{F(\mathcal{N}+n)}, \tag{30}$$

and rewrite the Eq.(29) in the following form

$$\begin{aligned} |\alpha\rangle &= C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(\frac{\mathcal{N}}{F(\mathcal{N})} A^\dagger\right)^n \|0\rangle \\ &= C_0 \exp\left(\frac{\alpha\mathcal{N}}{F(\mathcal{N})} A^\dagger\right) \|0\rangle. \end{aligned} \tag{31}$$

The operator

$$D(\alpha) \equiv C_0 \exp\left(\frac{\alpha\mathcal{N}}{F(\mathcal{N})} A^\dagger\right) \tag{32}$$

is the coherent displacement-operator in the form of the ordinary exponential function.

Let us remark that two displacement operators $\exp_F(\alpha A^\dagger)$ and $\exp\left(\frac{\alpha\mathcal{N}}{F(\mathcal{N})} A^\dagger\right)$ are essentially different, although they give rise to the same coherent states by acting on the vacuum state.

We would like to mention that the coherent state (31) is the same as that given in [10]. Indeed, one can easily show that

$$\left[A, \frac{\mathcal{N}}{F(\mathcal{N})} A^\dagger\right] = 1 \tag{33}$$

and therefore

$$[A,\ D(\alpha)] = \alpha D(\alpha), \tag{34}$$

which leads to Eq.(31) immediately.

The next example is the square realization of $su(1,1)$, (9). In this case $(\mathcal{N}/F(\mathcal{N}))A^\dagger = \frac{1}{N-1}(a^\dagger)^2$ in the sector S_0 and $(\mathcal{N}/F(\mathcal{N}))A^\dagger = \frac{1}{N}(a^\dagger)^2$ in the sector S_1. Then we obtain the well-known two-component coherent states of $su(1,1)$:

$$\begin{aligned}
|\alpha'\rangle_0 &= C_0 \sum_{k=0}^{\infty} \frac{{\alpha'}^{2k}}{\sqrt{(2k)!}} |2k\rangle, \qquad\qquad \alpha' = \sqrt{2}\alpha, \\
|\alpha'\rangle_1 &= C_0' \sum_{k=0}^{\infty} \frac{{\alpha'}^{2k+1}}{\sqrt{(2k+1)!}} |2k+1\rangle, \quad C_0' = C_0 {\alpha'}^{-1}.
\end{aligned} \tag{35}$$

Note that in (35) we use the usual Fock states.

It should be mentioned that the coherent states of the q-deformed oscillators ($m = 1$ case) can also be obtained by inserting (13) and (14) into the general formula (31). We have advanced the understanding of the problem on two points, that is, (i) the displacement-operator is expressed by the usual exponential form [16], not the so-called q-deformed exponential function [11, 16]; (ii) the q-deformed oscillator admits the multi-component squeezed and coherent states through its multiphoton realization (14) but the relationship is broken by the vacuum expectation value. The above presentation can also be applied to the case of the quantum algebra $su(1,1)_q$.

3.3 Squeezed states

We have constructed the coherent displacement-operator $D(\alpha)$ and the squeeze operator $S(z)$. For the oscillator, we know that the state $D(\alpha)S(z)|0\rangle$

$$\begin{aligned}
|\alpha, z\rangle \equiv D(\alpha)S(z)|0\rangle &= C_0 e^{\alpha a^\dagger} e^{\frac{z}{2} a^{\dagger 2}} |0\rangle \\
&\underline{\underline{\text{normalization}}} \quad e^{\alpha a^\dagger - \alpha^* a} e^{\frac{z}{2} a^{\dagger 2} - \frac{z^*}{2} a^2} |0\rangle
\end{aligned} \tag{36}$$

is just the squeezed state, which is also an eigenstate of $\mu a + \nu a^\dagger$. However, for the general case here, the state $D(\alpha)S(z)\|0\rangle$ are not the squeezed state equivalent to the ladder-operator definition, namely, it is not an eigenstate of $\mu A + \nu A^\dagger$. The coherent displacement-operator $D(\alpha)$ is a *good* operator in the sense that it enjoys the following property

$$D(-\alpha) A D(\alpha) = A + \alpha, \tag{37}$$

which can be easily obtained from the following identity

$$e^{\xi P} Q e^{-\xi P} = Q + \xi [P, Q] + \frac{\xi^2}{2!} [P, [P, Q]] + \cdots. \tag{38}$$

However, the squeeze operator $S(z)$ does not keep the Holstein-Primakoff/Bogoliubov transformation, namely

$$S^{-1}(z) A S(z) \neq \mu A + \nu A^\dagger. \tag{39}$$

This is why the state $D(\alpha)S(z)\|0\rangle$ is not an eigenstate of $\mu A + \nu A^\dagger$, as argued in the paper [9].

Now let us consider another special case, $\nu = -\beta^2$ or $z = \mu^{-1}\beta^2$. In this case it is easy to see that the coefficient C_n can be written as $C_n = \widehat{C}(n, \mu)\beta^n$ and the $\widehat{C}(n, \mu)$ satisfying

the following relation

$$\begin{aligned}\widehat{C}(n+1,\mu) &= \frac{1}{\mu}\sqrt{\frac{F(n)}{F(n+1)}}\widehat{C}(n-1,\mu)+\frac{1}{\mu\sqrt{F(n+1)}}\widehat{C}(n,\mu),\\ \widehat{C}(1,\mu) &= \left(\mu\sqrt{F(1)}\right)^{-1}\widehat{C}(0,\mu),\end{aligned}\tag{40}$$

where $\widehat{C}(0,\mu)\equiv C_0$ is a constant independent of μ. Now we introduce a new symbol by

$$\widehat{D}(n,\mu)\equiv\widehat{C}(n,\mu)\sqrt{F(n)!},\tag{41}$$

then $\widehat{D}(n,\mu)$ is determined by

$$\begin{aligned}&\widehat{D}(n+1,\mu)=\mu^{-1}\left(\widehat{D}(n,\mu)+F(n)\widehat{D}(n-1,\mu)\right),\\ &\widehat{D}(1,\mu)=\mu^{-1}C_0.\end{aligned}\tag{42}$$

The squeezed state $|\beta,\mu\rangle$ is obtained as

$$\begin{aligned}|\beta,\mu\rangle &= \sum_{n=0}^{\infty}\widehat{D}(n,\mu)\frac{\beta^n}{\sqrt{F(n)!}}\|n\rangle\\ &= \sum_{n=0}^{\infty}\widehat{D}(n,\mu)\frac{\beta^n}{F(n)!}(A^\dagger)^n\|0\rangle\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\beta\frac{\mathcal{N}}{F(\mathcal{N})}A^\dagger\right)^n\widehat{D}(\mathcal{N}+n,\mu)\|0\rangle\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\widehat{D}(\mathcal{N},\mu)\left(\beta\frac{\mathcal{N}}{F(\mathcal{N})}A^\dagger\right)^n\|0\rangle\\ &= \widehat{D}(\mathcal{N},\mu)\exp\left(\beta\frac{\mathcal{N}}{F(\mathcal{N})}A^\dagger\right)\|0\rangle\\ &= \widehat{D}(\mathcal{N},\mu)D(\beta)\|0\rangle.\end{aligned}\tag{43}$$

It is easy to show that the above infinite series converges for $|1/\mu|<1$ and $|\beta|<1$ under about the same assumptions on the asymptotic behavior of $f(x)$ as before. It is interesting to note for $F(n)=n^2$ and $\mu=1$, the recursion relations can be solved explicitly: $\widehat{D}(n,1)/C_0=n!$. In this case the creation part of the operator $\mu A+\nu A^\dagger$ ($\nu=-\beta^2$) can be considered as a small perturbation to the annihilation part.

Recall that for the oscillator, the squeezed state can also be written as

$$|\alpha,z\rangle=D(\alpha)S(z)|0\rangle\equiv S(\gamma)D(\alpha)|0\rangle,\tag{44}$$

where $\gamma=\alpha\cosh r-\alpha^*e^{i\theta}\sinh r$ ($z=re^{i\theta}$). The operator $\widehat{D}(\mathcal{N},\mu)$ seems to play the role of a squeeze operator, but not exactly. It contains only the number operator $\mathcal{N}$. In comparison with the oscillator case, one finds that $\widehat{D}(\mathcal{N},\mu)$ is obviously different from the squeeze operator $S(z)$.

Although the states $D(\alpha)S(z)\|0\rangle$ and $S(z)D(\alpha)\|0\rangle$ are not the eigenstates of $\mu A+\nu A^\dagger$, they might be important quantum states in quantum optics. Investigation of their non-classical properties will be of significance. We shall consider this problem in a separate paper.

4 Multimode generalization

The above formalism can be easily generalized to the multimode case. For simplicity we consider only the two-mode case [19]. Generalization to multimode is straightforward. Consider the two-mode photon field described by two independent modes

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \tag{45}$$

and introduce a two-mode multiphoton annihilation operator

$$A = f(N_1, N_2) a^m b^n, \tag{46}$$

where $N_1 = a^\dagger a$, $N_2 = b^\dagger b$, f is an arbitrary function with $f(n_1, n_2) \neq 0$ for n_1, n_2 non-negative integers. Note that f is not necessarily written as $f(N_1, N_2) = f_1(N_1) f_2(N_2)$. Then we have

$$\begin{aligned} &AA^\dagger = F(\mathcal{N}_1 + 1, \mathcal{N}_2 + 1), \quad A^\dagger A = F(\mathcal{N}_1, \mathcal{N}_2), \\ &[\mathcal{N}_i, A^\dagger] = A^\dagger, \quad [\mathcal{N}_i, A] = -A, \quad (i = 1, 2), \end{aligned} \tag{47}$$

where

$$\begin{aligned} \mathcal{N}_1 &\equiv \frac{1}{m}(N_1 - i), \quad \mathcal{N}_2 \equiv \frac{1}{n}(N_2 - j), \quad (0 \leq i \leq m-1, \ 0 \leq j \leq n-1), \\ F(\mathcal{N}_1 + 1, \mathcal{N}_2 + 1) &\equiv (N_1 + 1) \cdots (N_1 + m)(N_2 + 1) \cdots (N_2 + n) f(N_1, N_2) f^*(N_1, N_2) \\ &\equiv (m\mathcal{N}_1 + i + 1) \cdots (m\mathcal{N}_1 + i + m)(n\mathcal{N}_2 + j + 1) \\ &\quad \cdots (n\mathcal{N}_2 + j + n) f(m\mathcal{N}_1 + i, n\mathcal{N}_2 + j) f^*(m\mathcal{N}_1 + i, n\mathcal{N}_2 + j). \end{aligned} \tag{48}$$

This algebra is defined on a subspace $\bar{S}_{ij}$ of the sector S_{ij}. A convenient orthonormal basis of $\bar{S}_{ij}$ is given by $(k = 0, 1, 2, \cdots)$

$$\|k\rangle \equiv \frac{1}{\sqrt{F(k,k)!}} (A^\dagger)^k |i, j\rangle \propto |km + i, kn + j\rangle. \tag{49}$$

The representation on $\bar{S}_{ij}$ is

$$\begin{aligned} A^\dagger \|k\rangle &= \sqrt{F(k+1, k+1)} \|k+1\rangle, \\ A\|k\rangle &= \sqrt{F(k,k)} \|k-1\rangle, \\ \mathcal{N}_1 \|k\rangle &= \mathcal{N}_2 \|k\rangle = k\|k\rangle. \end{aligned} \tag{50}$$

We consider the eigenvalue equation

$$\left(\mu A + \nu A^\dagger\right) |\beta\rangle = \beta |\beta\rangle. \tag{51}$$

These states are degenerate. The degeneracy can be lifted by assuming that the $(m+n)$ photons are either created or annihilated together. This means the following conservation law

$$(\mathcal{N}_1 - \mathcal{N}_2) |\beta\rangle = 0. \tag{52}$$

In the representation (50) the condition is fulfilled automatically.

By identifying $F(k,k)$ here with $F(k)$ in section 2, the representation (50) takes the same form as (12). So, formally, the squeezed states can be investigated in the same manner as those in section 2.

5 Conclusion

So far we have described a displacement-operator formalism of the squeezed states for the general symmetry system. The coherent displacement-operator and the squeeze operator are explicitly constructed. Although these two operators are equivalent to the ladder-operator definition, their product does not bring us the squeezed states consistent with the ladder-operator definition, due to the lack of Holstein-Primakoff/Bogoliubov transformation. However, we can expect that the states $D(\alpha)S(z)\|0\rangle$ and $S(z)D(\alpha)\|0\rangle$ play important roles in quantum optics and it is a good challenge to study their non-classical properties.

As a special case of this formalism, we obtain the coherent displacement-operator of the q-deformed oscillator, which takes the form of the *usual exponential function* form.

This formalism can also be applied to the systems with self-similar potentials [20]. In these systems, the dynamical symmetry algebra (called q-ladder algebra) belongs to the general category (7). In the isospectral oscillator hamiltonian systems [21], the creation and the annihilation operators also enjoy the algebraic structure (7) and therefore their coherent states can be studied as a special case of this letter. In particular, by making use of the techniques in section 3, some interesting results on the relation of this coherent and squeezed states with those of the oscillator can be found. The concrete properties of the various coherent states arising from them will be published in a separate paper.

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